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QUASI-STATIC PROBLEM OF MECHANICS IN STRESSES OF A DEFORMABLE SOLID*

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The classical problem of mechanics in stresses of a deformable solid (problem B), including the variational one, is presented. The theorems of existence and uniqueness of a general solution of problem B and of Castigliano's maximum are proved, and the convergence of the successive approximations method is established when the respective linear problem has a unique solution. Methods of speeding up the convergence, including that of "rapid convergence" of successive approximations whose rate exceeds that of a geometric progression are considered. A new formulation of the quasi-static problem of mechanics in stresses of a deformable solid (problem A) is presented. It reduces to solving six equations for components of the stress tensor with six boundary conditions. The equivalence of problems A and B is proved. The respective variational formulation of problem A based on the introduction of some operator I is presented. The problem of general solution is defined. The theorems of uniqueness of solution of problem A, of maximum of operator I, and of uniqueness of that maximum are proved with certain constraints imposed on the determining equations.

1. Let in some Cartesian coordinate system the determining relations that bind the tensors of strain ε and of stress σ be specified in the form /1/

$$\varepsilon_{ij} = G_{ij}(\sigma) \tag{1.1}$$

We assume deformations to be small so that the Cauchy formulas relating these to the displacement vector $\mathbf{u}$

$$\varepsilon_{ij} = {}^1/_2 (u_{i,j} + u_{j,i}) \quad (\varepsilon = \operatorname{def} \mathbf{u}) \tag{1.2}$$

are satisfied.

Let the equations of the medium equilibrium be specified as

$$S_i \equiv \sigma_{ij,j} + X_i = 0 \tag{1.3}$$

where $\mathbf{X}$ are given volume forces, and the boundary conditions are of the mixed type: along the part Σ_1 of the body boundary displacements $\mathbf{u}^\circ$ are specified, and along part Σ_2 the loads S° are given by

$$u_i |_{\Sigma_1} = u_i^\circ, \quad \sigma_{ij} n_j |_{\Sigma_2} = S_i^\circ \tag{1.4}$$

We assume that all of the considered functions posess the smoothness necessary for applying the required transforms, and vary in the time interval $[0, t_1]$. A "natural state", is assumed prior to the time $t = 0$ i.e. the stress and strain tensors and their derivatives are then zero.

If the volume V occupied by the body is a simply connected region, the necessary and sufficient conditions of integrability of the system of differential equations (1.2) with respect to displacements are the St.-Venant equations which reduce to zero the symmetric incompatibility tensor η

$$\eta_{ij} \equiv \epsilon_{ikl} \epsilon_{jmn} \varepsilon_{kn,lm} = 0 \quad (\eta \equiv \operatorname{Ink} \varepsilon = 0) \tag{1.5}$$

In this case it is possible to express the displacement vector $\mathbf{u}$ in terms of strains ε_{ij}, and owing to formulas (1.1) also in terms of stresses (1.6). We call problem (1.3), (1.5), (1.1), (1.4) quasi-static (static) problem of mechanics of deformable solid (problem B). After the substitution of expressions (1.1) into Eqs. (1.5) we can write the latter as

$$\eta_{ij} \{\mathbf{G}(\sigma)\} = 0 \tag{1.6}$$

and the boundary conditions (1.4), after applying Cesaro's formulas /2/ and relations (1.1), as

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$$u_i\{G(\sigma)\}|_{\Sigma_1}=u_i^\circ,\quad \sigma_{ij}n_j|_{\Sigma_2}=S_i^\circ \tag{1.7}$$

Thus the equations that define problem B are of the form (1.6), (1.3), and (1.7).

The tensor function $\sigma\in T$ which for any smooth tensor function $\tau\in T_0$ satisfies the integral identity /3/

$$\int_V \varepsilon_{ij}(\sigma)\tau_{ij}\,dV=A_{\Sigma_1}(\sigma,\mathbf{u}^\circ) \tag{1.8}$$

is called the generalized solution of problem B. In the above identity $A_{\Sigma_1}(\sigma,\mathbf{u}^\circ)$ is the work of internal forces σ on a given displacement $\mathbf{u}^\circ$

$$A_{\Sigma_1}(\sigma,\mathbf{u}^\circ)=\int_{\Sigma_1}\sigma_{ij}n_ju_i^\circ\,d\Sigma$$

where $\tau\in T$ indicates that tensor τ satisfies conditions

$$\tau_{ij,j}+X_i=0,\quad \tau_{ij}n_j|_{\Sigma_2}=S_i^\circ \tag{1.9}$$

and $\tau\in T_0$ satisfies conditions

$$\tau_{ij,j}=0,\quad \tau_{ij}n_j|_{\Sigma_2}=0 \tag{1.10}$$

If the strain tensor is potential, i.e. there exists a scalar stress operator $w(\sigma)$ such that

$$\varepsilon_{ij}=G_{ij}(\sigma)=\frac{\partial w(\sigma)}{\partial\sigma_{ij}} \tag{1.11}$$

it is possible to introduce the Castiglianian K using formula /3/

$$K(\sigma)\equiv-\varphi(\sigma)+A_{\Sigma_1}(\sigma,\mathbf{u}^\circ),\quad \varphi(\sigma)\equiv\int_V w\,dV \tag{1.12}$$

In this case the problem of finding a general solution of problem B is equivalent to the determination of the "stationary point" of Castiglianian $K(\sigma)$ /3/

$$DK(\sigma,\tau)=0$$

2. Let us now assume that the determining relations (1.1) are fairly smooth.

Lemma 1. If the functional derivatives $\partial\varepsilon_{ij}(\sigma)/\partial\sigma_{kl}$ of the determining relations (1.1) exist, then the identity

$$\varphi(\sigma^{(2)})=\varphi(\sigma^{(1)})+A_{\Sigma_1}(\sigma^{(2)}-\sigma^{(1)},\mathbf{u}^\circ)+\frac{1}{2}\int_V\left[\frac{\partial\varepsilon_{ij}}{\partial\sigma_{kl}}\{\sigma^{(2)}+\eta(\sigma^{(2)}-\sigma^{(1)})\}(\sigma_{kl}^{(2)}-\sigma_{kl}^{(1)})(\sigma_{ij}^{(2)}-\sigma_{ij}^{(1)}\right)dV \tag{2.1}$$

is valid.

Indeed, introducing the function of the numerical argument $\xi\ (0\leqslant\xi\leqslant 1)$

$$f(\xi)\equiv\varphi\{\sigma^{(1)}+\xi(\sigma^{(2)}-\sigma^{(1)})\} \tag{2.2}$$

which on the indicated segment can be represented in the form

$$f(1)=f(0)+f'(0)+{}^1/_2\,f''(\eta),\ 0<\eta<1 \tag{2.3}$$

Then, substituting into (2.3) the expressions obtained from (2.2) and taking into account (1.12), we obtain

$$\varphi(\sigma^{(2)})=\varphi(\sigma^{(1)})+\int_V\varepsilon_{ij}(\sigma^{(1)})(\sigma_{ij}^{(2)}-\sigma_{ij}^{(1)})\,dV+{}^1/_2\int_V\frac{\partial\varepsilon_{ij}}{\partial\sigma_{kl}}\{\sigma^{(1)}+\eta(\sigma^{(2)}-\sigma^{(1)})\}(\sigma_{kl}^{(2)}-\sigma_{kl}^{(1)})(\sigma_{ij}^{(2)}-\sigma_{ij}^{(1)})\,dV \tag{2.4}$$

from which, taking into account (1.8), we have (2.1).

Theorem 2.1. (The maximum of Castiglianian). Assume that the determining equations (1.1) are such that for any symmetric second rank tensor $\mathbf{h}$ the inequality

$$\left[\frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} h_{kl}\right] h_{ij} \geqslant n_0 h_{ij} h_{ij}, \quad n_0 > 0 \tag{2.5}$$

is satisfied.

The stationary point of Castiglianian (1.10) is the point of maximum.

Indeed, setting in (2.1) $\sigma^{(2)} = \tau \in T_0$ and $\sigma^{(1)} = \sigma^*$ (the solution of problem B) and allowing for (2.5), we have

$$K(\tau) \equiv -\varphi(\tau) + A_{\Sigma_1}(\tau, u^0) \leqslant -\varphi(\sigma^*) \quad A_{\Sigma_1}(\sigma^*, u^0) - \frac{n_0}{2}\int_V (\tau_{ij} - \sigma_{ij}^*)(\tau_{ij} - \sigma_{ij}^*)\, dV \leqslant -\varphi(\sigma^*) + A_{\Sigma_1}(\sigma^*, u^0) \equiv K(\sigma^*) \tag{2.6}$$

Q.E.D.

Theorem 2.2 (of uniqueness). If conditions (2.5) are satisfied, there exists not more than one general solution of Problem B. Let us assume the opposite: there exist two solutions $\sigma^{(1)}$ and $\sigma^{(2)}$. Then it follows (1.8) that they satisfy the identity

$$\int_V [\varepsilon_{ij}(\sigma^{(2)}) - \varepsilon_{ij}(\sigma^{(1)})]\, \tau_{ij}\, dV = 0 \tag{2.7}$$

Moreover

$$[\varepsilon_{ij}(\sigma^{(2)}) - \varepsilon_{ij}(\sigma^{(1)})]\, \tau_{ij} = \int_0^1 \left[\frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} \{\sigma^{(1)} + \xi(\sigma^{(2)} - \sigma^{(1)})\} \times (\sigma_{kl}^{(2)} - \sigma_{kl}^{(1)})\, \tau_{ij}\, d\xi \tag{2.8}$$

Hence, setting in (2.8) $\tau_{ij} \equiv \sigma_{ij}^{(2)} - \sigma_{ij}^{(1)}$, from (2.5) we obtain

$$0 \geqslant \int_V [\varepsilon_{ij}(\sigma^{(2)}) - \varepsilon_{ij}(\sigma^{(1)})](\sigma_{ij}^{(2)} - \sigma_{ij}^{(1)})\, dV \geqslant n_0 \int_V (\sigma_{ij}^{(2)} - \sigma_{ij}^{(1)})(\sigma_{ij}^{(2)} - \sigma_{ij}^{(1)})\, dV$$

which implies that

$$\sigma_{ij}^{(2)} \equiv \sigma_{ij}^{(1)} \tag{2.9}$$

i.e. the uniqueness of solution of problem B.

Theorem 2.3. The Castiglianian has a unique maximum point.

Let $\sigma^{(1)}$ and $\sigma^{(2)}$ be two maximum points of the Castiglianian K. Then condition (2.7) must be satisfied at both of them and, by virtue of Theorem 2.2, formula (2.9) is valid.

Let us now consider some linear stress tensor operator

$$\pi_{ij} = \Pi_{ij}(\sigma) \tag{2.10}$$

such that in the functional space $\sigma \in T_0$ the quantity

$$(\sigma, \tau)_\pi \equiv \int_V \pi_{ij}(\sigma)\, \tau_{ij}\, dV$$

satisfies all axioms of the scalar product /4/ so that the considered functional space S is a Hilbert space. Let, moreover, operator (2.10) be such that the inequalities

$$n\pi_{ij}(\mathbf{h})\, h_{ij} \leqslant \left[\frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} h_{kl}\right] h_{ij} \leqslant N\pi_{ij}(\mathbf{h})\, h_{ij}, \quad 0 < n \leqslant N \tag{2.11}$$

are satisfied for any arbitrary symmetric tensor $\mathbf{h}$.

Note that when

$$\pi_{ij}(\sigma) \equiv {}^1/_2\, (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\, \sigma_{kl} = \sigma_{ij}$$

the first of inequalities (2.11) is equivalent to inequality (2.5) when $n = n_0$. For such operator $\mathbf{\Pi}$ we denote the Hilbert space S by S_0.

If there exists now a unique general solution of problem B in the case when $\mathbf{\Pi}$ (2.10) (problem $\mathbf{B}_\pi$) is the operator of determining relations (1.1), it is possible to arrange the method of successive approximations

$$\eta_{ij}\{\mathbf{\Pi}(\sigma^{(m+1)})\} = \eta_{ij}\{\mathbf{\Pi}(\sigma^{(m)})\} - \beta^{(m)}\eta_{ij}\{\mathbf{G}(\sigma^{(m)})\} \tag{2.12}$$

$$\sigma_{ij,j}^{(m+1)} + X_i = 0 \tag{2.13}$$

$$u_i\{\Pi(\sigma^{(m+1)})\}|_{\Sigma_1} = u_i\{\Pi(\sigma^{(m)})\}|_{\Sigma_1} - \beta^{(m)}[u_i\{G(\sigma^{(m)})\}|_{\Sigma_1} - u_i^\circ];\ \sigma_{ij}^{(m+1)}n_j|_{\Sigma_2} = S_i^\circ \tag{2.14}$$

beginning with some zero approximation $\sigma^{(0)}$ and setting $m = 0, 1, \ldots$

Theorem 2.4. Let there exist a unique general solution of problem B_π, conditions (2.11) be satisfied, and let the specified displacements satisfy the conditions

$$u^\circ \in L_p(\Sigma),\quad p > 4/3 \tag{2.15}$$

Let, moreover, the condition

$$[\varepsilon_{ij}(\sigma^{(0)}) - \pi_{ij}(\sigma^{(0)})]h_{ij} \leqslant n\pi_{ij}(\mathbf{h})h_{ij}$$

where $\mathbf{h}$ is an arbitrary symmetric tensor, be satisfied for the zero approximation $\sigma^{(0)}$.

Then there exists in some neighborhood

$$\|\sigma - \sigma^{(0)}\|_\pi \leqslant r$$

a general solution σ^* of problem B which is unique in that neighborhood and for any value of the iteration parameter $\beta \in (0, 2/N]$ the successive approximation process (2.12)—(2.14) converges to it beginning with $\sigma^{(0)}$, and

$$\|\sigma^{(m)} - \sigma^*\|_\pi \leqslant \frac{q^m}{1-q}\|\sigma^{(1)} - \sigma^{(0)}\|_\pi \tag{2.16}$$

$$q = \max(|1 - \beta n|,\ |1 - \beta N|) < 1$$

The proof follows from the analysis of the identity

$$\int_V \pi_{ij}(\sigma)\tau_{ij}\,dV = \int_V \pi_{ij}(\sigma)\tau_{ij} - \beta\Big[\int_V \varepsilon_{ij}(\sigma)\tau_{ij}\,dV - A_{\Sigma_1}(\tau, u^\circ)\Big] \tag{2.17}$$

and the application to it of the procedure used in /3/ for proving Theorem 3.1.

The process of successive approximations (2.12)— (2.14) thus converges at the rate of a geometric progression with denominator q whose minimum value $q = (N - n)/(N + n)$ is attained for $\beta = 2/(N + n)$.

Theorem 2.5. When the conditions of Theorem 2.4 are satisfied, operator $\varphi(\sigma^{(m)})$ converges to $\varphi(\sigma^*)$ and, consequently, Castiglianian $K(\sigma^{(m)})$ converges to Castiglianian $K(\sigma^*)$.

Indeed, setting in (2.6) $\tau = \sigma^{(m)}$ we obtain

$$\varphi(\sigma^{(m)}) - \varphi(\sigma^*) \leqslant A_{\Sigma_1}(\sigma^{(m)} - \sigma^*, u^\circ) + {}^1/_2 N\|\sigma^{(m)} - \sigma^*\|_\pi^2$$

Using Sobolev's theorems of imbedding /5/, we obtain from this for displacements u° that satisfy conditions (2.15)

$$\varphi(\sigma^{(m)}) - \varphi(\sigma^*) \leqslant (B + {}^1/_2 N)\|\sigma^{(m)} - \sigma^*\|_\pi^2$$

where B is some constant dependent only on region Σ_1. Hence, using (2.16) we obtain

$$\varphi(\sigma^{(m)}) - \varphi(\sigma^*) \leqslant (B + {}^1/_2 N)q^{2m}\|\sigma^{(0)} - \sigma^*\|_\pi > 0,\quad m \to \infty$$

To obtain a more rapid convergence than that of geometric progression it is necessary to impose constraints on the second functional derivatives of the determining relationships (1.1). Let the inequality

$$\left|\left[\frac{\partial^2\varepsilon_{ij}}{\partial\sigma_{kl}\partial\sigma_{mn}} h_{kl}h_{mn}\right]h_{ij}\right| \leqslant l(h_{ij}h_{ij})^{3/2},\quad l > 0 \tag{2.18}$$

be satisfied for any arbitrary symmetric tensor $\mathbf{h}$.

Let us further assume that the space S_1 with the introduced scalar product

$$(\sigma, \tau)_1 \equiv \int_V \left[\frac{\partial\varepsilon_{ij}}{\partial\sigma_{kl}}\sigma_{kl}\right]\tau_{ij}\,dV$$

is a Hilbert space for tensor functions $\tau \in T_0$ determined in the finite region V.

Theorem 2.6. (The rapid convergence method). Let the operator (2.10) be of the form

$$\pi_{ij}(\mathbf{h}) \equiv \frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} h_{kl}$$

and let there exist a unique general solution of the respective problem B_π. If the inequality (2.18) and the inequalities

$$n_1 h_{ij} h_{ij} \leqslant \left[\frac{\partial \varepsilon_{ij}}{\partial \sigma_{kl}} h_{kl}\right] h_{ij} \leqslant N_1 h_{ij} h_{ij}, \quad 0 \leqslant n_1 \leqslant N_1$$

are satisfied, and a is a positive integer such that

$$\int_V [\varepsilon_{ij}(\sigma^{(0)}) - \pi_{ij}(\sigma^{(0)})] \sigma_{ij}^{(0)} dV \leqslant n_1 a \int_V \sigma_{ij}^{(0)} \sigma_{ij}^{(0)} dV$$

then a number α, $0 < \alpha \leqslant 1$ can be found for which problem B has a unique solution σ^* in the neighborhood $\| \sigma^{(0)} - \sigma^* \|_1 \leqslant r_0$ when the inequality

$$q \leqslant a^{-\alpha} C, \quad q \equiv \frac{3}{2} \frac{l}{n_1} V^{\alpha/2}, \quad C \equiv \alpha (1 + \alpha)^{-(1+\alpha)/\alpha}$$

where r_0 is the smallest root of the equation

$$q r^{1+\alpha} - r + a = 0$$

is satisfied.

When $\beta = 1$, the successive approximation process beginning with $\sigma^{(0)}$ converges to that solution, and

$$\| \sigma^{(m)} - \sigma^* \|_1 \leqslant C_1 \delta^{(1+\alpha)^m}, \quad \delta \equiv C^{1/\alpha}, \quad C_1 \equiv \frac{a}{\delta (1 - \delta)}$$

Proof of this theorem is obtained from the analysis of identity (2.17) and the application to it of the procedure used in Sect.7 in /6/ for proving Theorem 1.

3. The problem of mechanics in stresses of deformable solids (problem A) was given a new formulation in /3/ that is a development of ideas expressed by Il'iushin /7/. Let us formulate this problem in another way. Consider some vector operator $\mathbf{R}$ of vector S (1.3) such that $\mathbf{R}(S) = 0$ only when $S = 0$. We form the combination of the deviator of the incompatibility tensor η with its spherical part mutiplied by the constant symmetric tensor ξ. We obtain

$$H_{ij} \equiv \Delta \varepsilon_{ij} + \theta_{,ij} - \varepsilon_{ik,kj} - \varepsilon_{jk,ki} + \xi_{ij} (\varepsilon_{kl,kl} - \Delta\theta) = 0 \tag{3.1}$$

Expressing in (3.1) the strains in terms of stresses using formulas (1.1), we compose six equations in six independent components of the stress tensor

$$H_{ij}(\sigma) + R_{i,j}(S) + R_{j,i}(S) - \xi_{ij} R_{k,k}(S) = 0 \tag{3.2}$$

Let the equilibrium conditions

$$S_i |_\Sigma \equiv (\sigma_{ij,j} + X_i) |_\Sigma = 0 \tag{3.3}$$

be satisfied at the boundary.

Problem A is now reduced to solving six Eqs. (3.2) with boundary conditions (3.3) and (1.7).

Theorem 3.1. Problem A is equivalent to problem B.

To prove it we contract Eqs. (3.2) with the unit tensor

$$(2 - \xi_{kk}) \; [\Delta\theta(\sigma) - \varepsilon_{ij,ij}(\sigma) + R_{k,k}(S)] = 0 \tag{3.4}$$

and apply to (3.2) the operator div

$$(\delta_{ij} - \xi_{ij}) \; [\Delta\theta(\sigma) - \varepsilon_{kl,kl}(\sigma) + R_{k,k}(S)]_{,j} = 0 \tag{3.5}$$

When $\xi_{kk} \neq 2$ we have from (3.5) and (3.4)

$$\Delta R_i(S) = 0$$

It follows from this the boundary condition (3.3), and the properties of operator $\mathbf{R}$ that

$S_i = 0$ throughout the domain V, i.e. the equilibrium equations (1.3) are satisfied. Hence it follows from (3.2) that $H_{ij} = 0$ and, consequently also $H_{kk} = 0$. But then the compatibility conditions (1.5) are satisfied, and the theorem is proved.

We introduce the notation

$$\varepsilon_{ij,k}\delta_{ij} \equiv \theta_k, \; \varepsilon_{ij,k}\delta_{jk} \equiv e_i$$
$$\sigma_{ij,k}\delta_{ij} \equiv p_k, \; \sigma_{ij,k}\delta_{jk} \equiv q_i$$

and consider the third rank tensor

$$E_{ijk} \equiv \varepsilon_{ij,k} + \delta_{ki}(^1/_2\theta_j - e_j) + \delta_{kj}(^1/_2\ \theta_i - e_i) + \xi_{ij}(e_k - \theta_k) + R_i(\mathbf{q})\delta_{jk} + R_j(\mathbf{q})\delta_{ik} - \xi_{ij}R_k(\mathbf{q}) \tag{3.6}$$

Equations (3.2) can then be written in the divergent form

$$E_{ijk,k} + Y_{ij} = 0 \tag{3.7}$$
$$Y_{ij} \equiv R_{i,j}(\mathrm{X}) + R_{j,i}(\mathrm{X}) - \xi_{ij}R_{k,k}(\mathrm{X})$$

Let the loads

$$\sigma_{ij}n_j|_\Sigma = S_i^\circ \tag{3.8}$$

and equilibrium conditions (3.3)

$$q_i|_\Sigma = -X_i|_\Sigma \tag{3.9}$$

be specified at the boundary Σ of the body.

Problem A then consists of solving Eqs. (3.7) with boundary conditions (3.8) and (3.9).

We shall now present the variational formulation of problem A For this we assume the existence of such scalar operator Ω dependent on stress gradients that the conditions

$$E_{ijk} = \partial\Omega/\partial\sigma_{ij,k} \tag{3.10}$$

for tensor (3.6) to be potential are satisfied.

We call the second rank χ symmetric tensor

$$\chi_{ij} \equiv E_{ijk}n_k \tag{3.11}$$

determinate on surface Σ, the tensor of streams.

We define operator I by the formula

$$I \equiv \int_V (\Omega - Y_{ij}\sigma_{ij})\,dV - \int_\Sigma \chi_{ij}\sigma_{ij}\,d\Sigma + \int_V \left[\frac{1}{2}(Aq_iq_i + B\sigma_{ij}n_j\sigma_{ik}n_k) + AX_iq_i - BS_i^\circ\sigma_{ij}n_j\right]d\Sigma \tag{3.12}$$

where A and B are some nonzero dimensional constants.

Theorem 3.2. At the equilibrium position operator (3.12) has a constant value

$$DI(\sigma, \delta\sigma) = 0 \tag{3.13}$$

Note that in (3.13) the streams χ are not varied (are assumed "frozen"), and that their expressions in formula (3.11) are substituted in it.

Using formula (3.13) and the Ostrogradskii—Gauss (*) theorem we obtain

$$\int_V (E_{ijk,k} + Y_{ij})\,\delta\sigma_{ij}\,dV = A\int_\Sigma (q_i + X_i)\,\delta q_i\,d\Sigma + B\int_\Sigma (\sigma_{ij}n_j - S_i^\circ)\,\delta\sigma_{ik}n_k\,d\Sigma \tag{3.14}$$

Owing to the arbitrariness of variations, from (3.14) we have Eqs.(3.7) and boundary conditions (3.8) and (3.9).

4. We define the general solution of problem A by the symmetric tensor σ that satisfies for any smooth symmetric tensor τ the integral identity

$$\int_V E_{ijk}(\sigma)\,\tau_{ij,k}\,dV + \int_\Sigma (Aq_i\tau_{ij,j} + B\sigma_{ij}n_j\tau_{ik}n_k)\,d\Sigma = N(\tau) \tag{4.1}$$

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*) Editor's Note: English equivalent is "Gauss divergence theorem".

where

$$N \equiv N^V + N_1^\Sigma + N_2^\Sigma \tag{4.2}$$

$$N^V(\tau) \equiv \int_V Y_{ij}\tau_{ij}\,dV;\; N_1^\Sigma(\tau) \equiv \int_\Sigma \chi_{ij}(\sigma)\,\tau_{ij}\,d\Sigma; \tag{4.3}$$

$$N_2^\Sigma(\tau) \equiv \int_\Sigma (BS_i^\circ \tau_{ik}n_k - AX_i\tau_{ik,k})\,d\Sigma$$

We introduce the notation

$$f = f_V + f_\Sigma,\; f_V \equiv \int_V \Omega\,dV,\; f_\Sigma \equiv \frac{1}{2}\int_\Sigma (Aq_iq_i + B\sigma_{ij}n_j\sigma_{ik}n_k)\,d\Sigma$$

Then, obviously

$$I \equiv f - N(\sigma) \tag{4.4}$$

and the integral identity (4.1) can be written in the form

$$Df(\sigma, \tau) = N(\tau)$$

This shows that the definition of the general solution of problem A is the same as the weak solution of that problem (i.e. the solution of the variational equation (3.13)).

Lemma 2. When the functional derivatives $\partial E_{ijk}(\sigma)/\partial\sigma_{lm,n}$ exist, then the identity

$$f(\sigma^{(2)}) = f(\sigma^{(1)}) + N(\sigma^{(2)} - \sigma^{(1)}) + \frac{1}{2}\int_V \frac{\partial E_{ijk}}{\partial\sigma_{lm,n}}\{\sigma^{(1)} + \eta(\sigma^{(2)} - \sigma^{(1)})\}\,[\sigma_{lm,n}^{(2)} - \sigma_{lm,n}^{(1)}]\,[\sigma_{ij,k}^{(2)} - \sigma_{ij,k}^{(1)}]\,dV \tag{4.5}$$

is valid. Indeed, by introducing function

$$\varphi(\xi) \equiv f\;\{\sigma^{(1)} + \xi(\sigma^{(2)} - \sigma^{(1)})\} \tag{4.6}$$

of the numerical argument ξ $(0 \leqslant \xi \leqslant 1)$ which admits on the indicated segment the representation (2.3), after the substitution in the expression of type (2.3) of quantities from (4.6) and using (3.10) and (4.4), we obtain

$$f(\sigma^{(2)}) = f(\sigma^{(1)}) + \int_V E_{ijk}^{(1)}(\sigma_{ij,k}^{(2)} - \sigma_{ij,k}^{(1)})\,dV + A\int_\Sigma q_i^{(1)}(q_i^{(2)} - q_i^{(1)})\,d\Sigma +$$

$$\frac{1}{2}\int_V \frac{\partial E_{ijk}}{\partial\sigma_{lm,n}}\{\sigma^{(1)} + \eta(\sigma^{(2)} - \sigma^{(1)})\}\,[\sigma_{lm,n}^{(2)} - \sigma_{lm,n}^{(1)}]\,\times$$

$$[\sigma_{ij,k}^{(2)} - \sigma_{ij,k}^{(1)}]\,dV + B\int_\Sigma \sigma_{ij}^{(1)}n_j(\sigma_{ik}^{(2)} - \sigma_{ik}^{(1)})\,n_k\,d\Sigma$$

from which, taking into account (4.2) and (4.3), we have (4.5).

Theorem 4.1. Assume that for any third rank tensor $\mathbf{h}$ which is symmetric with respect to the first two indices, the inequality

$$\left[\frac{\partial E_{ijk}}{\partial\sigma_{lm,n}}h_{lmn}\right]h_{ijk} \geqslant k_0h_{ijk}h_{ijk},\quad k_0 > 0 \tag{4.7}$$

is satisfied, and the stationary point of operator I (3.12) has a minimum.

Indeed, setting in identity (4.5) $\sigma^{(2)} = \tau$ and $\sigma^{(1)} = \sigma^*$, where σ^* is the solution of problem A, and taking into account (4.7), we have

$$I(\tau) = f(\tau) - N(\tau) \geqslant f(\sigma^*) - N(\sigma^*) + \frac{1}{2}k_0\int_V (\tau_{ij,k} - \sigma_{ij,k}^*)(\tau_{ij,k} - \sigma_{ij,k}^*)\,dV \geqslant f(\sigma^*) - N(\sigma^*) = I(\sigma^*)$$

Q.E.D.

Theorem 4.2. When conditions (4.7) are satisfied, problem A has not more than one general solution.

Let us assume the opposite: there are two solutions $\sigma^{(1)}$ and $\sigma^{(2)}$ which by virtue of (3.7) must satisfy the identity

$$\int_V [E_{ijk}(\sigma^{(2)}) - E_{ijk}(\sigma^{(1)})]\tau_{ij,k}\,dV = 0 \tag{4.8}$$

for any symmetric tensor τ.

The integrand in (4.8) may be written in the form

$$\int_0^1 \frac{\partial E_{ijk}}{\partial \sigma_{lm,n}} \{\sigma^{(1)} + \xi(\sigma^{(2)} - \sigma^{(1)})[\sigma^{(2)}_{lm,n} - \sigma^{(1)}_{lm,n}]\tau_{ij,k}\,d\xi\}$$

Setting here $\tau_{ij,k} \equiv \sigma^{(2)}_{ij,k} - \sigma^{(1)}_{ij,k}$, from (4.7) we obtain

$$0 \geqslant \int_V (E^{(2)}_{ijk} - E^{(1)}_{ijk})(\sigma^{(2)}_{ij,k} - \sigma^{(1)}_{ij,k})\,dV \geqslant k_0 \int_V (\sigma^{(2)}_{ij,k} - \sigma^{(1)}_{ij,k})(\sigma^{(2)}_{ij,k} - \sigma^{(1)}_{ij,k})\,dV$$

It follows from this that $\sigma^{(2)}_{ij,k} = \sigma^{(1)}_{ij,k}$, i.e. tensor $\sigma^{(2)}$ and $\sigma^{(1)}$ differ from each other only by a constant tensor. However by virtue of boundary conditions (3.8) that constant tensor is zero. This proves the uniqueness of solution of problem A.

Theorem 4.3. The point of minimum of operator I (3.12) is unique.

Let $\sigma^{(1)}$ and $\sigma^{(2)}$ be two points of minimum of operator I. Then conditions (4.8) are satisfied, and by virtue of Theorem 4.2 $\sigma^{(1)} = \sigma^{(2)}$.

5. Methods of successive approximations for solving problem A can be constructed similarly to the method set forth in Sect.2.

REFERENCES

1. POBEDRIA B.E., Lectures on Tensor Analysis. Moscow, Izd. MGU, 1979.

2. NOVATSKII V., Theory of Elasticity. Moscow, MIR, 1975.

3. POBEDRIA B.E., Some general theorems of the mechanics of a deformable solid. PMM, Vol.43, No.3, 1979.

4. LIUSTERNIK L.A. and SOBOLEV V.I., Elements of Functional Analysis. Moscow, NAUKA, 1965.

5. SOBOLEV S.L., Certain Applications of Functional Analysis in Mathematical Physics. English translation, Providence, American Mathematical Society, Vol. 7, 1963.

6. POBEDRIA B.E., The mathematical theory of nonlinear viscoelasticity. In: Elasticity and Nonelasticity, No.3. Moscow, Izd. MGU, 1973.

7. ILIUSHIN A.A., Plasticity. Fundamentals of the General Mathematical Theory. Moscow, Izd. Akad. Nauk. SSSR, 1963.

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